O(d,d)-invariance in inhomogeneous string cosmologies with perfect fluid

Jacques Demaret[1] and Elisa Di Pietro[2]

Institute of Astrophysics and Geophysics

Group of Theoretical Cosmology

University of Liège

B-4000 LIEGE-BELGIUM

Abstract

In the first part of the present paper, we show that $O(d,d)$-invariance usually known in a homogeneous cosmological background written in terms of proper time can be extended to backgrounds depending on one or several coordinates (which may be any space-like or time-like coordinate(s)). In all cases, the presence of a perfect fluid is taken into account and the equivalent duality transformation in Einstein frame is explicitly given. In the second part, we present several concrete applications to some four-dimensional metrics, including inhomogeneous ones, which illustrate the different duality transformations discussed in the first part. Note that most of the dual solutions given here do not seem to be known in the literature.

Key words: superstring cosmology - duality - exact solutions - pre-Big Bang.

[1]E-mail: demaret@astro.ulg.ac.be
[2]E-mail: dipietro@astro.ulg.ac.be

1 Introduction.

As it is well known, the standard cosmological model describes remarkably well most of the present Universe's properties: extragalactic sources redshift, light elements nucleosynthesis, the cosmic microwave background at 2.735 K, ... In spite of these successes, there are some problems that this model cannot solve: initial singularity, horizon problem, isotropy, flatness and dimensionality of the present Universe, ... Much work has been done in order to solve these problems, in particular in the framework of inflationary models (see e.g. [1] and [2] and the references therein), but these models have in general to be fine tuned in order to give expected results.

Some years ago appeared string cosmology [3]. We do not consider here string theory as issued from particle physics and quantum field theory in its full scope, but we restrict rather to its low-energy approximation, i.e. string effective theory viewed as an alternative theory to general relativity. String cosmology is characterized by the corresponding field equations which generalize Einstein's field equations. This theory contains a metric tensor (as in general relativity), a scalar field, called *dilatonic field*, and a rank-three antisymmetric tensor field, called *axionic field*. String theory has motivated the study of cosmological models because of its most remarkable property of symmetry called *T-duality* (it scale factor duality in the framework of effective theory) [4]. This symmetry allows, from a string theory solution, to construct a second solution, called *dual solution*, so that in string theory, the first solution is used to describe the "*post-Big Bang phase*", i.e. the Universe's evolution between the initial singularity until today, whereas the dual solution describes an eventual "*pre-Big Bang phase*" [5], i.e. the Universe as it could be before the Big Bang. Though there is no satisfying model explaining the transition between pre- and post-Big Bang, many cosmologists think that duality symmetry and in particular the pre-Big Bang phase could solve the singularity problem.

Duality symmetry associates with the present Universe (decelerated expanding Universe) an accelerated expanding Universe [6], so that the pre-Big Bang solution appears naturally as an inflationary phase and can in principle solve the isotropy and flatness problems. Furthermore, string theory is initially a 10 or 26-dimensional theory, depending on whether heterotic or bosonic strings are considered, but at low energy, this theory allows the compactification of some of these dimensions and the expansion of the others. So one can hope to explain the present dimensionality of the Universe by the compactification of the extra dimensions of string theory.

As one can see, one can expect a lot from string theory and in particular from the pre-Big Bang phase that appears naturally in this theory. Most of the work done so far on duality has been based on Meissner *et al.*'s paper [4]. The authors present

a duality transformation that can be applied to a homogeneous solution written in proper time only. Our aim is to show that duality symmetry can be applied to every kind of cosmological string solution. We shall show more particularly how to generalize Meissner *et al.*'s transformation so that it can be applied to:

- a homogeneous solution written in terms of a time coordinate that can be different from proper time. Indeed sometimes field equations cannot be solved in terms of proper time so that cosmological solutions are known in terms of another time coordinate and the integration allowing the passage from this time coordinate to proper time is not always possible. So if we want to build the dual of such a solution, it is necessary to extend Meissner *et al.*'s transformation to any time-like coordinate.

- a solution depending on one space coordinate. This could be the case for a spherically symmetric black hole solution. As one can hope that there exists a possibility of avoiding the Big Bang singularity in the framework of the pre-Big Bang scenario, one can imagine to overpass the black hole singularity using a similar mechanism.

- a more general solution that can depend on several coordinates. Indeed, we know that near the cosmological singularity, our Universe is neither isotropic nor homogeneous. Accordingly more general models have to be used to study the primordial Universe.

In all the preceding cases, we have supposed that space-time is filled with a perfect fluid. Moreover, as some authors prefer to consider Einstein frame as the physical frame, we have also developed the duality transformation directly in Einstein frame.

The present paper is organized as follows: in the first section, we establish all the theoretical duality transformations in string as well as in Einstein frames. First we write explicitly the duality transformation in the two frames in the case of fields depending on one coordinate which may be different from a time-like coordinate. But when this coordinate is time-like, our duality transformation is valid for any time coordinate used: proper time, logarithmic time, conformal time, ... Then we extend our results to the most general case of fields depending on several coordinates. For each case, duality transformation of the energy-momentum tensor is explicitly displayed.

In the second section, we apply our theoretical duality transformation to some concrete examples. Note that most of the dual solutions presented here are not known in the literature. Each of these examples has been chosen to illustrate a specific

aspect of duality transformation. Duality is performed on solutions written in terms of proper time and of logarithmic time, corresponding to four-dimensional homogeneous and inhomogeneous metrics. Most of these solutions are given in presence of a perfect fluid as well in both frames. In the last section, we briefly comment on the possibility of analyzing geometrical and physical properties of the dual solution obtained in section 3 in view of their eventual application to the pre-Big Bang scenario.

Note that the explicit form of all the relations presented in this paper (duality transformations, solutions, dual solutions, ...) has been checked by using symbolic programming (Reduce, Excalc, Mathematica, ...).

2 O(d,d)-invariance of string theory.

2.1 Duality when all fields depend on one coordinate only (space-like or time-like).

i. In the string frame.

At low energy, the tree-level effective action for closed superstring theory (in its bosonic sector) can be written as

$$S_{eff} = \frac{1}{2\kappa_D^2} \int \left(e^{-\phi} \left[R + [\nabla\phi]^2 - \frac{1}{12} H_{\alpha\beta\delta} H^{\alpha\beta\delta} \right] + L_m \right) \sqrt{\mid det\, g \mid}\; d^D x \qquad (1)$$

where

D	is the space-time dimension,
κ_D^2	is a parameter connected with the fundamental string length,
$det\, g$	is the determinant of the metric tensor $g_{\alpha\beta}$,
R	is the curvature scalar,
ϕ	is the massless dilatonic scalar field,
$H_{\alpha\beta\delta}$	is the completely antisymmetric tensor field strength defined by $H = dB$, where B is a rank-two antisymmetric tensor,
L_m	is the Lagrangian for the matter (perfect fluid),
$[\nabla\phi]^2$	stands for $g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$, where ∇_α is the covariant derivative with respect to x^α.

By varying the action with respect to $g_{\alpha\beta}$, ϕ and $B_{\alpha\beta}$, we can find respectively the following field equations[3] [9]

$$R^{\alpha}{}_{\beta} - \frac{1}{4} H^{\alpha\mu\nu}\, H_{\beta\mu\nu} + g^{\alpha\delta} \nabla_{\delta} \nabla_{\beta}\, \phi = \kappa_D^2\, e^{\phi}\, T^{(m)\alpha}{}_{\beta} \tag{2}$$

$$R + 2\,\Box\,\phi - [\nabla\phi]^2 - \frac{1}{12} H^2 = 0 \tag{3}$$

$$\nabla_{\mu}\, \left(e^{-\phi}\, H^{\mu\alpha\beta} \right) = 0 \tag{4}$$

where $T^{(m)}_{\alpha\beta}$ is the energy-momentum tensor derived from the Lagrangian L_m and $\Box$ stands for the dalembertian operator.

As we shall see, the action (1) is invariant under a symmetry transformation called "*O(d,d)-invariance*" (T-duality) where d is the number of coordinates the metric and other fields do not depend on. This invariance is well known for a particular homogeneous cosmological background without matter, i.e. for a metric $g_{\alpha\beta}$, a dilaton ϕ and a potential $B_{\alpha\beta}$ depending on time only and for $T^{(m)}_{\alpha\beta} = 0$ (see e.g. [4]). We shall first extend preceding work [4] to the case of fields depending on one coordinate only (which may be different from the time-like coordinate) in the presence of matter. Then, we shall show how the duality transformation can be generalized when all fields ($g_{\alpha\beta}$, ϕ, $T^{(m)}_{\alpha\beta}$ and $B_{\alpha\beta}$) depend on several coordinates.

Let us first consider the case of $g_{\alpha\beta}$, ϕ, $T^{(m)}_{\alpha\beta}$ and $B_{\alpha\beta}$ depending on one coordinate only that can be eventually different from the time-like coordinate. For simplicity, we shall note the coordinates $x^0, ..., x^d$ where $d = D - 1$ and order them so that we can say that the fields depend on x^0 and do not depend on x^1, ..., x^d. Note that 0-index does not refer necessary to the usual time-like coordinate but to the coordinate the fields depend on, i.e. x^0. We shall also, as in [4], introduce the following assumptions

$$g_{0i} = 0 \qquad\qquad B_{0i} = 0 \tag{5}$$

with (i = 1, ..., d), so we can write

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 \\ 0 & G \end{pmatrix} \qquad\qquad B_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \tag{6}$$

where B and G are $d \times d$ matrices and g_{00} is the metric's component relative to the x^0 coordinate; B_{ij}, G_{ij} $(i, j = 1, ..., d)$ and g_{00} are functions of x^0.

[3] Greek indices run from 0 to $D - 1$.

Defining respectively the "*shifted dilaton*" Φ and the $d \times d$ matrix M

$$\Phi = \phi - ln\sqrt{\mid det\, g \mid} \tag{7}$$

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{8}$$

we can rewrite the effective action (1) under the form

$$S_{eff} = \frac{1}{2\kappa_D^2} \int e^{-\Phi} \left(g^{00} \left[\frac{1}{8} Tr \left[\partial_0 M \eta \partial_0 M \eta \right] + \left[\partial_0 \Phi \right]^2 \right] \right.$$

$$\left. - \partial_0^2 g^{00} + \frac{1}{4}\, g_{00} \left[g^{00} \right]^2 + e^{\phi}\, L_m \right) d^D x \tag{9}$$

where $\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ is an element of the O(d,d)-group and I stands for the $d \times d$ unit matrix[4].

It is easy to see that the part of action (9) which does not include matter is invariant under [4]

- the following transformation on M: $M \rightarrow \bar{M} = \Omega\, M \Omega^T$

  where Ω is an element of the O(d,d)-group, such as $\Omega^T \eta \Omega = \eta$. Taking as usually $\Omega \equiv \eta$, we can write

  $$\bar{M} = \begin{pmatrix} \bar{G}^{-1} & -\bar{G}^{-1}\,\bar{B} \\ \bar{B}\,\bar{G}^{-1} & \bar{G} - \bar{B}\,\bar{G}^{-1}\,\bar{B} \end{pmatrix} = \begin{pmatrix} G - B\,G^{-1}\,B & B\,G^{-1} \\ -\,G^{-1}\,B & G^{-1} \end{pmatrix} \tag{10}$$

  So equation (10) gives the duality transformation for G and B

  $$\left\{ \begin{array}{l} G \rightarrow \bar{G} = \left(G\, - B\, G^{-1}\, B \right)^{-1} \\ B \rightarrow \bar{B} = -\, G^{-1}\, B\, \left(G\, - B\, G^{-1}\, B \right)^{-1} \end{array} \right. \tag{11}$$

- the following transformation on Φ: $\Phi \rightarrow \bar{\Phi} = \Phi$

  Using equation (7), we can rewrite this transformation in terms of ϕ and $\bar{\phi}$:

  $$\phi \rightarrow \bar{\phi} = \phi + \frac{1}{2}\, ln \left(\frac{det\, \bar{G}}{det\, G} \right) \tag{12}$$

---

[4] When x^0 is the proper time, we are back to the hypothesis used by Meissner *et al.* in [4]. It is easy to see that in this case, our action (9) is the same as theirs.

- the following transformation on g_{00}:

$$g_{00} \rightarrow \bar{g}_{00} = g_{00} \tag{13}$$

The transformations (11)-(13) can be compared with Busher's transformations ([7], [8]) obtained when hypothesis (5) is not realized and in absence of matter and potential B. Busher has shown that, starting from a solution with $B = 0$ and $g_{0i} \neq 0$, a torsion potential B is generated after duality.

It remains now to see how the energy-momentum tensor must transform if we want that the complete action (9) stays invariant under duality. If we introduce transformations (11)-(13) into equation (2) without making any change in the energy-momentum tensor[5], we transform equation (2) in

$$R^{\alpha}{}_{\beta} - \frac{1}{4} H^{\alpha\mu\nu}\, H_{\beta\mu\nu} + g^{\alpha\delta} \nabla_{\delta} \nabla_{\beta}\, \phi = -\kappa_D^2\, e^{\phi} \sqrt{\frac{det\, \bar{G}}{det\, G}}\, T^{(m)\alpha}{}_{\beta} \tag{14}$$

if $\alpha \neq 0$, and in

$$R^{\alpha}{}_{\beta} - \frac{1}{4} H^{\alpha\mu\nu}\, H_{\beta\mu\nu} + g^{\alpha\delta} \nabla_{\delta} \nabla_{\beta}\, \phi = \kappa_D^2\, e^{\phi} \sqrt{\frac{det\, \bar{G}}{det\, G}}\, T^{(m)\alpha}{}_{\beta} \tag{15}$$

if $\alpha = 0$, where it is important to keep in mind that 0-index corresponds to the coordinate the metric and other fields depend on, so that $T^{(m)\,0}{}_{0}$ is not necessary the energy density.

Thus, we can say that:

- if $\alpha \neq 0$, $T^{(m)\alpha}{}_{\beta}$ must transform as follows

$$T^{(m)\alpha}{}_{\beta} \rightarrow \bar{T}^{(m)\alpha}{}_{\beta} = -\left[\frac{det\, \bar{G}}{det\, G}\right]^{-1/2} T^{(m)\alpha}{}_{\beta} \tag{16}$$

- if $\alpha = 0$, $T^{(m)\alpha}_{\beta}$ must transform as follows

$$T^{(m)\alpha}{}_{\beta} \rightarrow \bar{T}^{(m)\alpha}{}_{\beta} = \left[\frac{det\, \bar{G}}{det\, G}\right]^{-1/2} T^{(m)\alpha}{}_{\beta} \tag{17}$$

[5]In this paper, we shall always consider a diagonal energy-momentum tensor $T^{(m)\alpha}{}_{\beta}$.

In conclusion, the duality transformation in string frame is given by the relations (11)-(13), (16)-(17). We shall later on present a few concrete examples of application of this transformation for a few four-dimensional space-times.

ii. In the Einstein frame.

It is sometimes easier to solve the field equations when they are written in Einstein frame, so that, in some cases, it is more convenient to work in Einstein frame. We shall see that starting from a solution written in the Einstein frame, it is possible to obtain directly the dual solution in the same frame[6].

The Einstein frame can be built from the string frame by the following conformal transformation[7]

$$\tilde{g}_{\alpha\beta} = e^{-\phi} g_{\alpha\beta} \tag{18}$$

If we introduce this transformation in the action (1), we obtain the same action, i.e. the same theory, written in the Einstein frame:

$$S_{eff} = \frac{1}{2\kappa_D^2} \int \left(\tilde{R} - \frac{1}{12} e^{-2\phi}\, \tilde{H}_{\alpha\beta\delta}\, \tilde{H}^{\alpha\beta\delta} - \frac{1}{2} \left[\tilde{\nabla}\phi \right]^2 + \tilde{L}_m \right) \sqrt{|\, det\, \tilde{g}\, |}\; d^D x \tag{19}$$

where $\tilde{L}_m = e^{2\phi}\, L_m$ and where, by convention, $H_{\alpha\beta\delta} = \tilde{H}_{\alpha\beta\delta}$ so that, for the dual field, $\bar{H}_{\alpha\beta\delta} = \tilde{\bar{H}}_{\alpha\beta\delta}$.

The variations of action (19) with respect to $\tilde{g}_{\alpha\beta}$, ϕ and $\tilde{B}_{\alpha\beta}$ give respectively the following field equations [9][8]:

$$\tilde{R}_{\alpha\beta} - \frac{1}{2}\tilde{g}_{\alpha\beta}\tilde{R} = \kappa_D^2\, \tilde{T}^{(m)}_{\alpha\beta} + \kappa_D^2\, \tilde{T}^{(\phi)}_{\alpha\beta} + \kappa_D^2\, \tilde{T}^{(H)}_{\alpha\beta} \tag{20}$$

$$\frac{1}{6} e^{-2\phi}\, \tilde{H}^{\alpha\beta\delta}\, \tilde{H}_{\alpha\beta\delta} + \tilde{\Box}\, \phi - \tilde{T}^{(m)} = 0 \tag{21}$$

$$\tilde{\nabla}_\mu \left[e^{-2\phi}\, \tilde{H}^{\mu\alpha\beta} \right] = 0 \tag{22}$$

where $\tilde{T}^{(m)}$ is $T^{(m)}_{\alpha\beta}$'s trace and the different contributions to the energy-momentum tensor are defined by

[6] Of course, it is possible, starting from a solution given in Einstein frame, to write the corresponding solution in string frame, via the conformal transformation (18) and to find in this way the dual solution in string frame using (11)-(13), (16)-(17). Starting from the latter, we can find the dual solution in Einstein frame via the inverse conformal transformation. Of course, it is easier to apply directly duality in Einstein frame.

[7] In what follows, the tilded variables refer to the Einstein frame whereas the "barred" variables are related to the dual solution.

[8] Note that in Copeland *et al.*'s paper [9], a 1/2 factor should be ignored in the last term of the l.h.s. of equation (2.11).

$$
\begin{aligned}
\kappa_D^2 \tilde{T}^{(m)}_{\alpha\beta} &= e^{2\phi}\, \kappa_D^2\, T^{(m)}_{\alpha\beta} \\
\kappa_D^2 \tilde{T}^{(\phi)}_{\alpha\beta} &= \tfrac{1}{2}\left(\tilde{\nabla}_\alpha \phi\, \tilde{\nabla}_\beta \phi - \tfrac{1}{2}\tilde{g}_{\alpha\beta}\left[\tilde{\nabla}\phi\right]^2\right) \\
\kappa_D^2 \tilde{T}^{(H)}_{\alpha\beta} &= \tfrac{1}{4} e^{-2\,\phi}\left(\tilde{H}_{\alpha\mu\sigma}\tilde{H}_\beta{}^{\mu\sigma} - \tfrac{1}{6}\tilde{g}_{\alpha\beta}\tilde{H}^{\alpha\beta\delta}\tilde{H}_{\alpha\beta\delta}\right)
\end{aligned}
$$

In order to build up the duality transformation in the Einstein frame, we have to apply the conformal transformation to the duality transformation in the string frame, i.e. to the duality transformation written in terms of G, B and $T^{(m)\alpha}{}_\beta$, and given by (11)-(13), (16)-(17). We obtain in this way

$$
\begin{aligned}
\tilde{G} \rightarrow &\quad \bar{\tilde{G}} = q\, e^{-2\phi}\, \tilde{P}^{-1} \\
\phi \rightarrow &\quad \bar{\phi} = \phi - lnq \\
\tilde{g}_{00} \rightarrow &\quad \bar{\tilde{g}}_{00} = q\, \tilde{g}_{00} \\
\tilde{B} \rightarrow &\quad \bar{\tilde{B}} = -e^{-2\phi}\, \tilde{G}^{-1}\, \tilde{B}\, \tilde{P}^{-1} \\
\tilde{T}^{(m)\alpha}{}_\beta \rightarrow &\quad \bar{\tilde{T}}^{(m)\alpha}{}_\beta = \pm\, q\, \tilde{T}^{(m)\alpha}{}_\beta
\end{aligned}
\tag{23}
$$

where we have introduced the $d \times d$ matrix $\tilde{P}$ and the number q defined by

$$
\begin{aligned}
\tilde{P} &= \tilde{G} - e^{-2\phi}\, \tilde{B}\, \tilde{G}^{-1}\, \tilde{B} \\
q &= e^{d\phi}\sqrt{det\, \tilde{G}\tilde{P}}
\end{aligned}
\tag{24}
$$

and where we must take the "+" sign in $\bar{\tilde{T}}^{(m)\alpha}{}_\beta$ if $\alpha = 0$ and the "$-$" sign if $\alpha \neq 0$. We see that in Einstein frame, because of the presence of the conformal factor in the metric's definition (18), $\tilde{g}_{00}$ will be modified by duality.

2.2 What happens when the fields depend on several coordinates ?

Consider now a more general case i.e. all fields ($g_{\alpha\beta}$, ϕ, $B_{\alpha\beta}$ and $T^{(m)}_{\alpha\beta}$) depending on several coordinates. Again, we shall order the coordinates so that we can say that all fields depend on x^0, ..., x^{D-d-1} and do not depend on x^{D-d}, ..., x^{D-1}. It is again necessary to suppose that the metric and the potential can be written as

$$
g(x^k) = \left(\begin{array}{cccccc|c}
g_{00} & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & g_{11} & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & g_{D-d-1,D-d-1} & 0 \\
\hline
 & & & 0 & & & G
\end{array}\right)
\tag{25}
$$

$$B(x^k) = \left(\begin{array}{c|c} \tilde{0} & 0 \\ \hline 0 & B \end{array} \right) \tag{26}$$

where G and B are $d \times d$ matrices whose components depend on x^k $(k = 0, ..., D - d - 1)$ and where $\tilde{0}$ is the $D - d \times D - d$ zero matrix.

In the same way as before, we can show that, using definitions (7) and (8), the action (1) can be rewritten as

$$\begin{aligned} S_{eff} = \frac{1}{2\kappa_D^2} \int e^{-\Phi} & \left(e^{\phi} L_m + \sum_{i=0}^{D-d-1} g^{ii} \left\{ [\partial_i \Phi]^2 + \frac{1}{8} Tr \left[\partial_i M \eta \partial_i M \eta \right] \right\} \right. \\ & \left. - \sum_{i=0}^{D-d-1} \left\{ \partial_i^2 g^{ii} + \frac{1}{4} g^{ii} \left[\partial_i g^{ii} \partial_i g_{ii} - \sum_{j \neq i} \partial_i g^{jj} \partial_i g_{jj} \right] \right\} \right) d^D x \end{aligned} \tag{27}$$

Again, it is easy to see that the action (27) is invariant under duality transformation given by (11)-(13) and (16)-(17), since duality does not act on the g^{ii}'s components (i=0,...,D-d-1). So, duality transformation has the same form when all fields depend on one or several coordinates, the only difference being the dimension of M and η matrices. In the Einstein frame, the duality transformation is again given by the transformations (23) with definitions (24).

2.3 A simple case: B=0.

Since later we shall apply duality transformation to simple examples for which $B = 0$, it is interesting to examine the form of the duality transformation in this case.

We have seen that independently of the number of coordinates the metric tensor and the other fields depend on, the duality transformation is given by (11)-(13) and (16)-(17), in string frame, and by (23)-(24) in Einstein frame. Accordingly, we shall give the form of this transformation when $B = 0$, without taking into account the number of coordinates all fields depend on.

If we introduce $B = 0$ in the matrix M defined by (8), we obtain the following simple form for M:

$$M = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \tag{28}$$

If we use transformation (10), we obtain for the matrix $\bar{M}$

$$\bar{M} = \begin{pmatrix} \bar{G}^{-1} & 0 \\ 0 & \bar{G} \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \tag{29}$$

so the duality transformation in string frame takes the form

$$\begin{aligned} G \rightarrow &\quad \bar{G} = G^{-1} \\ \phi \rightarrow &\quad \bar{\phi} = \phi - \, ln\,(det\, G) \\ B = 0 \rightarrow &\quad \bar{B} = 0 \\ T^{(m)\alpha}{}_{\beta} \rightarrow &\quad \bar{T}^{(m)\alpha}{}_{\beta} = \pm \, [det\, G]\; T^{(m)\alpha}{}_{\beta} \end{aligned} \tag{30}$$

where we must take the "+" sign in the energy-momentum tensor if $\alpha = i$ and the "$-$" sign if $\alpha \neq i$, the i-index corresponding to all the coordinates the metric and the other fields depend on $(i = 0, ..., D - d - 1)$.

In Einstein frame, the duality transformation obtained by taking $B = 0$ is

$$\begin{aligned} \tilde{G} \rightarrow &\quad \bar{\tilde{G}} = q\, e^{-2\phi}\, \tilde{G}^{-1} \\ \phi \rightarrow &\quad \bar{\phi} = \phi - lnq \\ \tilde{g}_{00} \rightarrow &\quad \bar{\tilde{g}}_{00} = q\, \tilde{g}_{00} \\ \tilde{B} = 0 \rightarrow &\quad \bar{\tilde{B}} = 0 \\ \tilde{T}^{(m)\alpha}{}_{\beta} \rightarrow &\quad \bar{\tilde{T}}^{(m)\alpha}{}_{\beta} = \pm\, q\, \tilde{T}^{(m)\alpha}{}_{\beta} \end{aligned} \tag{31}$$

with $q = e^{d\phi} \mid det\, \tilde{G} \mid$ and where we must again take the "+" sign in the energy-momentum tensor if $\alpha = i$ and the "$-$" sign, if $\alpha \neq i$ $(i = 0, ..., D - d - 1)$.

3 Examples

3.1 Flat FLRW metric in general relativity.

As a first example, we shall consider the flat FLRW metric, i.e. Einstein-de Sitter metric, that can be written as

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right] \tag{32}$$

where t is the proper time and a is the scale factor function of t. The solution of general relativity field equations in presence of perfect fluid with state equation $p = \rho/3$ (radiation) is given by[9]

$$\begin{aligned} a(t) &\quad = t^{1/2} \\ \kappa^2\, \rho(t) &\quad = 3/4\, t^{-2} \\ \kappa^2\, p(t) &\quad = 1/4\, t^{-2} \end{aligned} \tag{33}$$

[9] In all the following examples, $D = 4$ so that $\kappa_D^2 = \kappa_4^2$. But, for simplicity, we shall note κ^2 instead of κ_4^2.

It is important to note that one recovers the general relativity limit from (2)-(4) only when $H = 0$, $\phi = 0$ and $T^{(m)} = 0$, where $T^{(m)}$ is the energy-momentum tensor's trace, so that in this case only can a solution of Einstein field equations with a perfect fluid be at the same time a solution of string field equations. So (33) being a general relativity solution with $T^{(m)} = 0$, is also solution of string theory field equations in string as well as in Einstein frames. This metric is homogeneous and written in terms of proper time ($g_{00} = -1$) so our duality transformation in string frame given by (11)-(13) and (16)-(17) reduces to the one given in [4]. The matrix G can be written as

$$G = \begin{pmatrix} a^2(t) & 0 & 0 \\ 0 & a^2(t) & 0 \\ 0 & 0 & a^2(t) \end{pmatrix} \tag{34}$$

Introducing this matrix and (33) in (30), we find the corresponding dual solution in string frame:

$$\begin{aligned} \bar{a}(t) &= t^{-1/2} \\ \bar{\phi}(t) &= -3\,ln(t) \\ \kappa^2\,\bar{\rho}(t) &= 3/4\,t \\ \kappa^2\,\bar{p}(t) &= -1/4\,t \end{aligned} \tag{35}$$

where the dual metric is

$$ds^2 = -dt^2 + \bar{a}(t)^2 \left[dx^2 + dy^2 + dz^2\right] \tag{36}$$

To find the dual solution in Einstein frame, one only has to apply transformation (31) to (33) and (34) (as we are in general relativity, we have $\tilde{G} = G$). So the dual solution in Einstein frame is

$$\begin{aligned} \bar{\tilde{a}}(t) &= t \\ \bar{\tilde{g_{00}}}(t) &= -t^3 \\ \bar{\phi}(t) &= -3\,lnt \\ \kappa^2\,\bar{\rho}(t) &= 3/4\,t \\ \kappa^2\,\bar{p}(t) &= -1/4\,t \end{aligned} \tag{37}$$

The dual metric found in this way is

$$ds^2 = -t^3 dt^2 + \bar{\tilde{a}}^2(t) \left[dx^2 + dy^2 + dz^2\right] \tag{38}$$

We see that, in the Einstein frame, starting from a solution written in terms of proper time, the dual solution is no more expressed in terms of proper time. If we want to write it in terms of proper time, it is necessary to change the time-like coordinate from t to $\tilde{t}$ so that $\tilde{t}$ defined by

$$\tilde{t} = \int t^{3/2}\,dt \propto t^{5/2} \tag{39}$$

is the new proper time in Einstein frame. We can finally write the dual solution in terms of proper time:

- the dual metric
$$ds^2 = -d\tilde{t}^2 + \bar{\tilde{a}}^2 \left[dx^2 + dy^2 + dz^2\right] \tag{40}$$
with $\bar{\tilde{a}}(\tilde{t}) = \tilde{t}^{\,2/5}$

- the dual fields
$$\begin{aligned} \bar{\phi}(\tilde{t}) &= -6/5\, ln\tilde{t} \\ \kappa^2\, \bar{\rho}(\tilde{t}) &= 3/4\, \tilde{t}^{\,2/5} \\ \kappa^2\, \bar{p}(\tilde{t}) &= -1/4\, \tilde{t}^{\,2/5} \end{aligned} \tag{41}$$

Note that duality transformation applied to a solution written in the proper time of Einstein frame always implies a proper time redefinition.

3.2 Bianchi I metric in general relativity.

As a second example, we shall consider the Bianchi I cosmological solution of general relativity with perfect fluid [10].

The Bianchi I metric takes the following form

$$ds^2 = -(abc)^2 d\eta^2 + a^2 dx^2 + b^2 dy^2 + c^2 dz^2 \tag{42}$$

where a, b and c are the scale factors functions of η. We do not use in (42) proper time but logarithmic time denoted by η and related to proper time t by

$$dt = a\, b\, c\, d\eta \tag{43}$$

This metric describes a four-dimensional homogeneous cosmological background so x^0 is the time-like coordinate and the energy-momentum tensor can be written as $T^{(m)\alpha}{}_{\beta} = diag(-\rho, p_1, p_2, p_3)$[10] where $\rho(\eta)$ is the energy-density and p_1, p_2 and p_3 are the three pressure's components. We shall take for state equation of the perfect fluid $p = \gamma\rho$ where we shall restrict ourselves to $0 \leq \gamma < 1$.

It is important to remind that in section 2 duality transformation has been established without making any hypothesis about the time-like coordinate to be used. Indeed duality transformation can be applied for any function $g_{00}(x^0)$: it is not

[10] In all the following examples, we shall consider a diagonal energy-momentum tensor.

necessary that $g_{00} = -1$ as in the case of proper time.

The solution of general relativity field equations with a perfect fluid characterized by an isotropic pressure ($p = p_1 = p_2 = p_3$), i.e.

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = \kappa^2 T^{(m)}_{\alpha\beta} \tag{44}$$

is displayed below[11] [10]

$$a(\eta) = e^{(\beta_0+\delta_0)\,\eta/3}\, sinh^{-2/3/(1-\gamma)}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \tag{45}$$

$$b(\eta) = e^{(\delta_0-2\beta_0)\,\eta/3}\, sinh^{-2/3/(1-\gamma)}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \tag{46}$$

$$c(\eta) = e^{(\beta_0-2\delta_0)\,\eta/3}\, sinh^{-2/3/(1-\gamma)}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \tag{47}$$

$$\kappa^2\rho(\eta) = \frac{\epsilon}{3}\, sinh^{-2\omega}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \tag{48}$$

$$\kappa^2 p(\eta) = \frac{\gamma\epsilon}{3}\, sinh^{-2\omega}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \tag{49}$$

where ω, ϵ, β_0 and δ_0 are constants related by

$$\omega = \frac{\gamma+1}{\gamma-1} \tag{50}$$

and

$$\epsilon = \beta_0^2 + \delta_0^2 - \beta_0\delta_0 \tag{51}$$

The duality transformation can be applied to a string theory solution only. The above solution is also a string theory solution only when the energy-momentum tensor's trace is null, so when $p = \rho/3$. Indeed it can easily be checked that the solution obtained with $\gamma = 0$ (dust universe) is not a solution of equations (2)-(4) contrary to the solution with $\gamma = 1/3$. As a solution in explicit form for $\gamma = 1/3$ cannot be found in terms of proper time but only in terms of logarithmic time, as far as we know, we have to resort to our duality transformation given in the preceding section.

For $\gamma = 1/3$, the above solution takes the following form

$$a(\eta) = e^{(\beta_0+\delta_0)\eta/3}\, sinh^{-1}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{52}$$

[11] This solution is known in closed form in terms of proper time for $\gamma = 0$ only [11].

$$b(\eta) = e^{(\delta_0 - 2\beta_0)\eta/3}\, sinh^{-1}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{53}$$

$$c(\eta) = e^{(\beta_0 - 2\delta_0)\eta/3}\, sinh^{-1}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{54}$$

$$\kappa^2 \rho(\eta) = \frac{\epsilon}{3}\, sinh^4\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{55}$$

$$\kappa^2 p(\eta) = \frac{\epsilon}{9}\, sinh^4\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{56}$$

with

$$\epsilon = \beta_0^2 + \delta_0^2 - \beta_0\delta_0 \tag{57}$$

This solution is a string theory solution in string as well as in Einstein frames. As the metric is four-dimensional and homogeneous, the matrix G defined by (6) can be written as

$$G = \begin{pmatrix} a(\eta)^2 & 0 & 0 \\ 0 & b(\eta)^2 & 0 \\ 0 & 0 & c(\eta)^2 \end{pmatrix} \tag{58}$$

and using this matrix G, the relations (30) enable one to find the dual solution in string frame whereas the relations (31) with $\tilde{G} = G$ give the dual solution in Einstein frame, i.e. respectively

$$\bar{a}(\eta) = e^{-(\beta_0 + \delta_0)\eta/3}\, sinh\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{59}$$

$$\bar{b}(\eta) = e^{(2\beta_0 - \delta_0)\eta/3}\, sinh\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{60}$$

$$\bar{c}(\eta) = e^{(2\delta - \beta_0)\eta/3}\, sinh\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{61}$$

$$\bar{\bar{a}}(\eta) = e^{-(\beta_0 + \delta_0)\eta/3}\, sinh^{-2}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{62}$$

$$\bar{\bar{b}}(\eta) = e^{-(\delta_0 - 2\beta_0)\eta/3}\, sinh^{-2}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{63}$$

$$\bar{\bar{c}}(\eta) = e^{-(\beta_0 - 2\delta_0)\eta/3}\, sinh^{-2}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{64}$$

$$\bar{\phi}(\eta) = 6\, ln\left(sinh\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right]\right) \tag{65}$$

$$\kappa^2 \bar{\rho}(\eta) = \frac{\epsilon}{3}\, sinh^{-2}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{66}$$

$$\kappa^2\bar{p}(\eta) = -\frac{\epsilon}{9}\, sinh^{-2}\left[\frac{1}{3}\sqrt{\epsilon}\,\eta\right] \tag{67}$$

with

$$\epsilon = \beta_0^2 + \delta_0^2 - \beta_0\delta_0 \tag{68}$$

where the tilded scale factors are related to Einstein frame whereas the untilded ones refer to string frame.

As is manifest in this example, fluid's pressure changes sign after duality but not its density.

3.3 Bianchi I metric with a scalar field.

We shall consider again the metric (42) but we shall apply duality transformation to the string theory solution with $\phi \neq 0$. The solution with a perfect fluid with state equation $p = \gamma\rho$ (with $0 \leq \gamma < 1$) of equations (2)-(3) is given by [10]

- The scale factors in string frame

$$\left| \begin{array}{ll} a(\eta) & = e^{(D-AB)\,\eta/2} sinh^{k+\omega/2}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \\ b(\eta) & = e^{(D-AB-2\beta_0)\,\eta/2} sinh^{k+\omega/2}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \\ c(\eta) & = e^{(D-AB-2\delta_0)\,\eta/2} sinh^{k+\omega/2}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \end{array} \right. \tag{69}$$

- The scale factors in Einstein frame

$$\left| \begin{array}{ll} \tilde{a}(\eta) & = e^{-AB\eta/2} sinh^{k}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \\ \tilde{b}(\eta) & = e^{-(AB/2+\beta_0)\,\eta} sinh^{k}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \\ \tilde{c}(\eta) & = e^{-(AB/2+\delta_0)\,\eta} sinh^{k}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \end{array} \right. \tag{70}$$

- The other fields

$$\left| \begin{array}{rl} \phi(\eta) & = D\eta + \omega\, ln\left(sinh\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right]\right) \\ \kappa^2\rho(\eta) & = \epsilon\, A\, e^{E\eta} sinh^{l}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \\ \kappa^2 p(\eta) & = \epsilon\gamma\, A\, e^{E\eta} sinh^{l}\left[\frac{1}{2}\sqrt{\epsilon}\,(1-\gamma)\,\eta\right] \end{array} \right. \tag{71}$$

with

$$A = \left[3 - \left(\frac{3\gamma - 1}{1-\gamma}\right)^2\right]^{-1}$$

$$B = -2\left(\beta_0 + \delta_0 + \frac{\phi_0}{2}\,\frac{3\gamma-1}{1-\gamma}\right)$$
$$C = \beta_0\delta_0 - \frac{1}{4}\phi_0^2$$
$$D = \phi_0 - \frac{B}{A}\,\left(\frac{3\gamma-1}{1-\gamma}\right)$$
$$E = 2\left(\frac{B(1+3\gamma)}{2A(1-\gamma)} + \beta_0 + \delta_0 - \phi_0\right)$$
$$k = -\frac{2}{A(1-\gamma)}$$
$$\omega = \frac{-4}{A}\left(\frac{3\gamma-1}{(1-\gamma)^2}\right)$$
$$\epsilon = \frac{B^2}{4} - AC$$
$$l = -2 + \frac{4}{A}\,\frac{3\gamma+1}{(1-\gamma)^2}$$

Contrary to the previous example of Bianchi I metric in general relativity, we can apply here the duality transformation to the above solution for any $0 \leq \gamma < 1$: this is so because we are in string theory instead of in general relativity.

We shall give e.g. the explicit solution for dust fluid ($\gamma = 0$):

$$\begin{array}{l}
a(\eta) = e^{\phi_0\eta/2} \\
b(\eta) = e^{(\phi_0-2\beta_0)\,\eta/2} \\
c(\eta) = e^{(\phi_0-2\delta_0)\,\eta/2} \\
\tilde{a}(\eta) = e^{(\beta_0+\delta_0-\phi_0/2)\,\eta/2}\,sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\tilde{b}(\eta) = e^{(-\beta_0+\delta_0-\phi_0/2)\,\eta/2}\,sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\tilde{c}(\eta) = e^{(\beta_0-\delta_0-\phi_0/2)\,\eta/2}\,sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\phi(\eta) = (3\phi_0/2 - \beta_0 - \delta_0)\,\eta + 2\,ln\left(sinh\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right]\right) \\
\kappa^2\rho(\eta) = \frac{\epsilon}{2}\,e^{(\beta_0+\delta_0-3\phi_0/2)\,\eta} \\
\kappa^2 p(\eta) = 0
\end{array} \tag{72}$$

with as constraint

$$\epsilon = \beta_0^2 + \delta_0^2 - \phi_0\,(\beta_0+\delta_0) + \frac{3}{4}\phi_0^2 \tag{73}$$

The dual solution can be found by applying to the above solution transformation (30) in string frame and transformation (31) in Einstein frame. We obtain in this way

$$\begin{array}{l}
\bar{a}(\eta) = e^{-\phi_0 \eta/2} \\
\bar{b}(\eta) = e^{(\beta_0-\phi_0/2)\,\eta} \\
\bar{c}(\eta) = e^{(\delta_0-\phi_0/2)\,\eta} \\
\bar{\bar{a}}(\eta) = e^{-(\beta_0+\delta_0-\phi_0/2)\,\eta/2}\, sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\bar{\bar{b}}(\eta) = e^{(\beta_0-\delta_0+\phi_0/2)\,\eta/2}\, sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\bar{\bar{c}}(\eta) = e^{(-\beta_0+\delta_0+\phi_0/2)\,\eta/2}\, sinh^{-1}\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right] \\
\bar{\phi}(\eta) = (-3\phi_0/2+\beta_0+\delta_0)\,\eta + 2ln\left(sinh\left[\frac{1}{2}\sqrt{\epsilon}\,\eta\right]\right) \\
\kappa^2\bar{\rho}(\eta) = \frac{\epsilon}{2}\, e^{-(\beta_0+\delta_0-3\phi_0/2)\,\eta} \\
\kappa^2\bar{p}(\eta) = 0
\end{array} \tag{74}$$

with again (73) as constraint.

3.4 Texeira et al.'s inhomogeneous metric.

For the following example, we have chosen to perform duality on an inhomogeneous metric that has the following form [12]

$$ds^2 = -e^{2\nu(z)}\, dt^2 + z^2\left[dx^2+dy^2\right] + \frac{z}{F(z)}\, dz^2 \tag{75}$$

The exact solution of the corresponding general relativistic field equations for this space-time filled with a perfect fluid with state equation $p = \rho/3$ is well known [12]:

$$\begin{array}{ll}
\nu(z) & = -\frac{1}{2}ln(z) - \frac{1}{2}ln(6-z^5) \\
F(z) & = \frac{1}{5}\,(6-z^5)^3 \\
\kappa^2 p_1(z) & = \kappa^2 p_2(z) = \kappa^2 p_3(z) = z^2(6-z^5)^2 \\
\kappa^2\rho(z) & = 3\,z^2(6-z^5)^2
\end{array} \tag{76}$$

<u>Case 1.</u>

As the metric is four-dimensional and depends on z-coordinate only, the matrix G defined by (6) can be written as

$$G = \begin{pmatrix} -e^{2\nu(z)} & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix} \tag{77}$$

Using the transformation given by (30), we find the dual solution in string frame, under the following form

$$ds^2 = -e^{-2\nu(z)}\, dt^2 + \frac{1}{z^2}\left[dx^2 + dy^2\right] + \frac{z}{F(z)}\, dz^2 \tag{78}$$

The corresponding dual scalar fields can be written as

$$\begin{array}{rl} \bar{\phi}(z) & = -3\, ln(z) + ln(6 - z^5) \\ \kappa^2\, \bar{p}_1(z) & = -z^5(6 - z^5) \\ \kappa^2\, \bar{p}_2(z) & = -z^5(6 - z^5) \\ \kappa^2\, \bar{p}_3(z) & = z^5(6 - z^5) \\ \kappa^2\, \bar{\rho}(z) & = -3z^5(6 - z^5) \end{array} \tag{79}$$

Some remarks have to be done here. First we see that starting from a solution with isotropic pressure, we find a dual solution with anisotropic pressure: two components are negative and one remains positive (the one corresponding to the z-coordinate), but in absolute value, the pressure remains the same in the three directions. Secondly, we note that after performing the duality transformation, the energy density becomes negative and so we are led to question the physical validity of this dual solution.

<u>Case 2.</u>

It is interesting to note that as $B = 0$, we are free to choose G's dimension. Indeed, (77) is the matrix G with maximal dimension but as the potential B does not appear in the duality transformation, we may choose a matrix G with a smaller dimension. For example, we can write the metric (75) as

$$g_{\alpha\beta} = \left(\begin{array}{cc|c} z/F(z) & 0 & 0 \\ 0 & -e^{2\nu} & 0 \\ \hline 0 & 0 & G(z) \end{array}\right) \tag{80}$$

with

$$G = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix} \tag{81}$$

and we can perform the duality transformation on the solution (76) with (81) as matrix G. Again duality transformation is given by (30) in string frame and the dual solution obtained is

- for the metric:

$$ds^2 = -e^{2\nu(z)}\, dt^2 + \frac{1}{z^2}\left[dx^2 + dy^2\right] + \frac{z}{F(z)}\, dz^2 \tag{82}$$

- for the scalar fields:

$$
\begin{aligned}
\bar{\phi}(z) &= -4\, ln(z) \\
\kappa^2\, \bar{\rho}(z) &= 3\, z^6(6-z^5)^2 \\
\kappa^2\, \bar{p}_1(z) &= -z^6(6-z^5)^2 \\
\kappa^2\, \bar{p}_2(z) &= -z^6(6-z^5)^2 \\
\kappa^2\, \bar{p}_3(z) &= z^6(6-z^5)^2
\end{aligned}
\tag{83}
$$

As we have removed g_{00} from the matrix G given by (81) with respect to its form (77), we can see that the energy density remains positive. In fact, we can say in general that if g_{ii} is present in G, then the corresponding energy-momentum tensor's component $T^{(m)}_{ii}$ will change sign after duality while if g_{ii} is not included in G, then the corresponding $T^{(m)}_{ii}$ will keep the same sign after duality. Indeed, in the first case (G given by (77)), g_{00}, g_{11} and g_{22} are present in G, so ρ, p_1 and p_2 become negative after duality whereas p_3 remains positive while in the second case (G given by (81)), only g_{11} and g_{22} are present in G and so only p_1 and p_2 become negative.

In the same way, we can perform duality with a 1×1 matrix for G, for example:

$$G = \left(e^{2\nu} \right) \tag{84}$$

or

$$G = \left(z^2 \right) \tag{85}$$

The only way to keep an isotropic pressure after duality is to include all spatial metric components in G, i.e. in the case of a homogeneous metric[12].

3.5 Inhomogeneous Senovilla's metric.

Now we shall apply duality transformation to a metric depending on several coordinates, i. e. inhomogeneous Senovilla's metric given by [13]

$$ds^2 = e^{2f}(-dt^2 + dx^2) + h(qdy^2 + q^{-1}dz^2) \tag{86}$$

where f, h et q are functions of t and x. The solution found in general relativity in presence of a perfect fluid with state equation $p = \rho/3$ has been given in [13]:

$$
\begin{aligned}
e^{f(x,t)} &= cosh^2(at)\, cosh(3ax) \\
h(x,t) &= cosh(at)\, sinh(3ax)\, cosh^{-2/3}(3ax) \\
q(x,t) &= cosh^3(at)\, sinh(3ax)
\end{aligned}
\tag{87}
$$

[12] In fact, there is another mathematically possible case for which the pressure remains isotropic after duality, i.e. when none of the spatial metric components is in G, so that g_{00} is the only element of G. However, the energy density ρ becomes negative which casts some doubt on the physical validity of the corresponding dual solution.

$$
\begin{aligned}
\kappa^2\,\rho(x,t) &= 15a^2\,\left[cosh(at)\,cosh(3ax)\right]^{-4} \\
\kappa^2\,p(x,t) &= 5a^2\,\left[cosh(at)\,cosh(3ax)\right]^{-4}
\end{aligned}
\tag{88}
$$

where a is an integration constant.

As the metric depends on two coordinates, t and x, the matrix G can be written as

$$
G = \begin{pmatrix} hq & 0 \\ 0 & hq^{-1} \end{pmatrix}
\tag{89}
$$

Using relations given in (30), which constitute the duality transformation in string frame, we find the dual metric:

$$
ds^2 = e^{2f}(-dt^2 + dx^2) + h^{-1}(q^{-1}dy^2 + qdz^2)
\tag{90}
$$

with $f(x,t)$, $h(x,t)$ and $q(x,t)$ given by (87), and the dual scalar fields:

$$
\begin{aligned}
\bar{\phi}(x,t) =& \quad -2\,\left(ln\left[cosh(at)\right] + ln\left[sinh(3ax)\right] - \tfrac{2}{3}ln\left[cosh(3ax)\right]\right) \\
\kappa^2\,\bar{\rho}(x,t) =& \quad 15a^2h^2(x,t)\left[cosh(at)cosh(3ax)\right]^{-4} \\
\kappa^2\,\bar{p}_1(x,t) =& \quad 5a^2h^2(x,t)\left[cosh(at)cosh(3ax)\right]^{-4} \\
\kappa^2\,\bar{p}_2(x,t) =& \quad -5a^2h^2(x,t)\left[cosh(at)cosh(3ax)\right]^{-4} \\
\kappa^2\,\bar{p}_3(x,t) =& \quad -5a^2h^2(x,t)\left[cosh(at)cosh(3ax)\right]^{-4}
\end{aligned}
\tag{91}
$$

where the function $h(x,t)$ is given by (87).

We note that since only g_{22} and g_{33} are present in matrix G, only p_2 and p_3 change sign after duality while ρ and p_1 remain positive. If we apply transformation (31) to (86), we find the dual metric in Einstein frame:

$$
ds^2 = e^{2f}h^2\,(-dt^2 + dx^2) + h(q^{-1}dy^2 + qdz^2)
\tag{92}
$$

with $f(x,t)$, $h(x,t)$ and $q(x,t)$ given by (87), the dual scalar fields (91) being the same in the two frames.

3.6 Mars' inhomogeneous non-diagonal metric.

We shall now consider the inhomogeneous non-diagonal metric [14][13] given by

$$\begin{aligned} ds^2 = e^{h\,t} \Big(& e^{s\,r^2} \, cosh(2\,k\,t) \, \left[-dt^2 + dr^2 \right] + \\ & r^2 \, cosh(2\,k\,t) \, d\varphi^2 + \frac{1}{cosh(2\,k\,t)} \, (dz + k\,r^2 d\varphi)^2 \Big) \end{aligned} \tag{93}$$

where h, k and s are constants.
This metric is solution of string field equations (2)-(4) in the absence of perfect fluid with a dilatonic field given by

$$\phi = h\,t \tag{94}$$

with

$$h^2 + 4k^2 - 4s = 0 \tag{95}$$

as constraint. As the metric depends on the two coordinates t and r, the G matrix can be written as

$$G = \frac{e^{h\,t}}{cosh(2\,k\,t)} \begin{pmatrix} r^2 cosh^2(2\,k\,t) + k^2 r^4 & kr^2 \\ kr^2 & 1 \end{pmatrix} \tag{96}$$

In this case, B, p and ρ are absent and the duality transformation (30) takes the following form

$$\begin{array}{ll} G & \rightarrow \bar{G} = G^{-1} \\ \phi & \rightarrow \bar{\phi} = \phi - ln(det\,G) \end{array} \tag{97}$$

Using these relations, we can write for the matrix $\bar{G}$

$$\bar{G} = G^{-1} = \frac{e^{-h\,t}}{cosh(2\,k\,t)} \begin{pmatrix} r^{-2} & -k \\ & \\ -k & cosh^2(2\,k\,t) + k^2 r^2 \end{pmatrix} \tag{98}$$

so that the dual metric becomes

$$\begin{aligned} ds^2 = \quad & e^{ht + sr^2} \, cosh(2kt) \, [-dt^2 + dr^2] \\ & + \left[\left(r^{-1} \, d\varphi - krdz\right)^2 + cosh^2(2kt) \, dz^2 \right] \, \frac{e^{-ht}}{cosh(2kt)} \end{aligned} \tag{99}$$

Finally the dilatonic field can be written as

$$\bar{\phi} = -h\,t - 2\,lnr \tag{100}$$

the constraint (95) remaining still valid.

[13]In this paper, M.Mars does not introduce directly the dilatonic field: he works with a stiff fluid. But we know that a massless scalar field is equivalent to a stiff fluid: we can pass from the scalar field expression to the fluid expression by the relation: $p = \rho = \dot{\phi}^2/4$ [15].

3.7 Schwarzschild's metric in general relativity.

The last example we shall consider is Schwarzschild's metric in general relativity[14]:

$$ds^2 = -e^{\lambda(r)}dt^2 + e^{\nu(r)}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \tag{101}$$

with

$$e^{\lambda(r)} = e^{-\nu(r)} = 1 - \frac{2GM}{r} \tag{102}$$

This solution is given for $\phi = p = \rho = 0$ and is different from the preceding examples since the metric is now written in spherical coordinates. As this metric depends on two coordinates (r, θ), we could be tempted to choose the following matrix G

$$G = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & r^2 sin^2\theta \end{pmatrix} \tag{103}$$

If we apply the duality transformation (97) to the above matrix G, we realize that the dual solution obtained in this way is not a solution of field equations (2)-(4). The fundamental reason for this is due to the fact that the spatial metric is written in spherical coordinates. If we rewrite the metric in Cartesian coordinates, we see that, in fact, it depends on the three space coordinates. Thus the matrix G has in fact to be chosen as the one-dimensional matrix

$$G = \left(e^{\lambda}\right) \tag{104}$$

Using the duality transformation (97), we find the corresponding dual solution

$$\begin{aligned} \bar{G} &= \left(e^{-\lambda}\right) \\ \bar{\phi} &= -\lambda \end{aligned} \tag{105}$$

so that the genuine dual metric, back in spherical coordinates, can be written as

$$ds^2 = e^{-\lambda}\left[-dt^2 + dr^2\right] + r^2\left[d\theta^2 + sin^2\theta d\varphi^2\right] \tag{106}$$

Note that, in this way, we do not modify the angular components of the metric, so we keep the spherical symmetry of the initial Schwarzschild solution.

[14]As it is a vacuum solution, it is also a string theory solution and we can so perform on it the duality transformation.

4 Some future perspectives...

The main use of the dual transformation is to build new solutions of the string theory field equations which could serve as models of the pre-Big Bang, before the initial singularity, in the framework of Veneziano's pre-Big Bang cosmology.

A credible implementation of this scenario requires the use of realistic models for the primordial Universe, i.e. anisotropic and inhomogeneous models. Some models among those considered in section 3 could be analyzed in more detail: in particular, the geometrical and physical properties of the exact dual solutions explicitly built could be studied in view of examining the presence of a singularity and the existence of a pre-Big Bang inflationary phase.

Another important point to consider is : "Could there remain in the present Universe some relics from an eventually pre-Big Bang phase that could involve some observational evidence ?".

In the same way, the problem of the possible avoidance of the black hole singularity could be tackled using e.g. a black hole solution joined to its dual solution (see e.g. § 3.7) before reaching the singularity. This leaves the door open to a series of speculative idea: is it possible, in this framework, for the dual solution of a black hole to be an open door to another Universe ? In any case, this remains a challenge for our imagination...

Acknowledgements

This work was supported in part by Belgian Interuniversity Attraction Pole P4/O5 as well as by a grant from "Fonds National de la Recherche Scientifique".

References

[1] Coles, P., and Lucchin, F. (1995). *Cosmology. The Origin and Evolution of Cosmic Structure.*, chap.7, John Wiley & Sons, New York.

[2] Linde, A. (1994). *Lectures on inflationary cosmology*, hep-th/9410082.

[3] Gasperini, M., and Veneziano, G. (1993). *Astropart. Phys.* **1***, 317*; Veneziano, G. (1997). *A simple/short introduction to pre-Big Bang physics/cosmology*, lecture presented at Erice School of Subnuclear Physics, 35th Course "Highlights:

50 Years Later", Erice, Italy; Veneziano, G. (1996). *Status of string cosmology: basic concepts and main consequences* in *String gravity and physics at the Planck energy scale*, ed. N. Sanchez and A. Zichichi, Kluwer Academic Publishers; An update collection of papers on string theory is available at *http://www.to.infn.it∼gasperini/*.

[4] Meissner, K.A., and Veneziano, G. (1991). *Phys. Lett. B***267***, 33.*

[5] Gasperini, M., and Veneziano, G. (1992). *Phys. Lett. B***277***, 256*; Meissner, K.A., and Veneziano, G. (1991). *Mod. Phys. Lett.* **A6***, 3397.*

[6] Kar, S. (1997). *Phys. Lett. B***401***, 234.*

[7] Busher, T. (1988). *Phys. Lett. B***201***, 466*; *Phys. Lett. B***194***, 59.*

[8] Horowitz, G.T. (1993). *The Black Side of String Theory: Black Holes and Black Strings* in *String Theory and Quantum Gravity*, ed. J. Harvey et al., World Scientific, hep-th/9210119.

[9] Copeland, E.J., Lahiri, A., and Wands, D. (1994). *Phys. Rev. D***50***, 4868.*

[10] Di Pietro, E. (1998). *Exact cosmological solutions of string theory, Internal report*, University of Lige.

[11] Jacobs, K.C. (1968). *Astrophys. J.* **153***, 661.*

[12] Texeira, A.F. da F., Wolk, I., Som, M.M. (1977). *J. Phys. A***10***, 1679.*

[13] Senovilla, J.M.M. (1990). *Phys. Rev. Lett.***64***, 2219.*

[14] Mars, M. (1995). *Phys. Rev. D***51***, R3989.*

[15] Wainwright, J., Ince, W.C.W., and Marshman, B.J. (1979). *Gen. Rel. Grav.* **10***, 259.*